Poisson方程式, Laplaus方程式：2階線型微分方程式

Laplaus

(i) $\phi = f(r)$ が解 $\Longrightarrow \phi = \lambda f(r)$ も解 (λ：定数)

(ii) $\phi = f(r),\ \phi = g(r)$ が解 $\Longrightarrow \phi = f(r) + g(r)$ も解

(iii) f, g が $\Delta\phi = 0$ の独立な解 $\Longrightarrow \Delta\phi = 0$ の一般解は $\lambda f + \mu g$

Poisson

(iv) $\left\{\begin{array}{l} f \text{ を } \Delta\phi = 0 \text{ の一般解} \\ g \text{ を } \Delta\phi = \underset{\neq 0}{\alpha} \text{ の一般解} \end{array}\right\} \Longrightarrow \Delta\phi = \alpha$ の一般解は、$f + g$

$r \geq a$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\phi = 0$$

$$\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\phi = 0$$

$$r^2\frac{d}{dr}\phi = \text{const.} = \lambda$$

$$\frac{d\phi}{dr} = \frac{\lambda}{r^2}$$

$$\therefore \phi = -\frac{\lambda}{r} + \mu \quad (\mu : \text{const.})$$

$r \leq a$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\phi = -\frac{1}{\varepsilon_0}\rho$$

$$\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\phi = -\frac{1}{\varepsilon_0}\rho r^2$$

$$r^2\frac{d}{dr}\phi = -\frac{1}{3\varepsilon_0}\rho r^3 + K \quad (K : \text{const.})$$

$$\frac{d}{dr}\phi = -\frac{1}{3\varepsilon_0}\rho r + \frac{K}{r^2}$$

$$\therefore \phi = -\frac{1}{6\varepsilon_0}\rho r^2 \underbrace{- \frac{K}{r} + \nu}_{\text{Laplaus eq. の一般解}} \quad (\nu : \text{const.})$$

Poisson eq. の一般解

λ, μ, K, ν の求め方：境界条件

(1) $r \to \infty$ で $\phi \to 0$ （ポテンシャルの原点を無限遠にとる。）

$\hookrightarrow \mu = 0$

(2) $r = 0$ で ϕ は有限（電荷分布が全ての点で有限）

$\hookrightarrow K = 0$

$$\therefore \begin{cases} r > a & \phi = -\dfrac{\lambda}{r} \\ r < a & \phi = -\dfrac{1}{6\varepsilon_0}\rho r^2 + \nu \end{cases}$$

(3) $r = a$ で、$\phi_{外} = \phi_{内}$

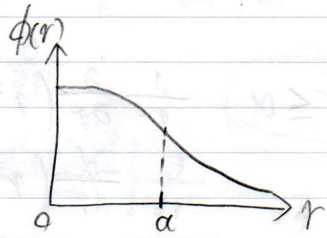
$$\boxed{-\frac{\lambda}{a} = -\frac{1}{6\varepsilon_0}\rho a^2 + \nu}$$

(4) $r = a$ で $E_{外} = E_{内}$

$$\left(\mathbb{E} = -\mathrm{grad}\,\phi = -\left\{ \frac{\partial \phi}{\partial r}\mathbb{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\mathbb{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \varphi}\mathbb{e}_\varphi \right\} \right)$$

$$\boxed{\frac{\lambda}{a^2} = -\frac{1}{3\varepsilon_0}\rho a}$$

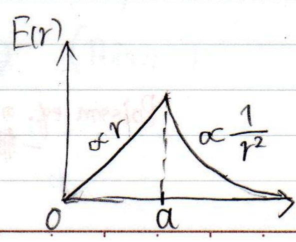
$$\begin{cases} \lambda = -\dfrac{\rho}{3\varepsilon_0}a^3 \\ \nu = \dfrac{\rho}{2\varepsilon_0}a^2 \end{cases}$$



よって
$$\begin{cases} r \geq a \text{ で } & \phi(r) = \dfrac{\rho}{3\varepsilon_0}a^3\dfrac{1}{r} \\ r \leq a \text{ で、} & \phi(r) = -\dfrac{\rho}{6\varepsilon_0}r^2 + \dfrac{\rho}{2\varepsilon_0}a^2 \end{cases}$$

$E(r) = -\dfrac{\partial \phi(r)}{\partial r}$ より、

$$\begin{cases} r \geq a \text{ で } & E(r) = \dfrac{\rho}{3\varepsilon_0}a^3\dfrac{1}{r^2} \\ r \leq a \text{ で } & E(r) = \dfrac{\rho}{3\varepsilon_0}r \end{cases}$$



（Dirichlet 問題：境界で ϕ の値を与える）