まとめ.

静電場の求め方(真空中)

(1) Coulomb の法則から直接求める.

$$\mathbb{E}(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')}{|r-r'|^3}(r-r')\,d^3r'$$

$dx'dy'dz' = dv$ (体積積分)

(2) 積分形の Gauss の法則を使う.

$$\int_{C上} \mathbb{E}\cdot d\mathbb{S} = \frac{1}{\varepsilon_0}(C内の全電荷)$$

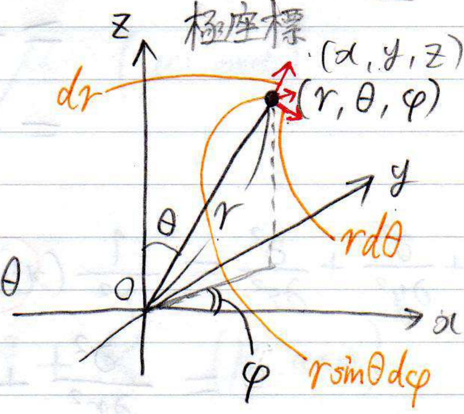
(3) $\begin{cases} \mathrm{div}\,\mathbb{E} = \frac{1}{\varepsilon_0}\rho(r) \\ \mathrm{rot}\,\mathbb{E} = 0 \end{cases}$ を用いる.　(boundary condition, B.C.)

ただし、解くべき問題に応じて、境界条件を設定する.

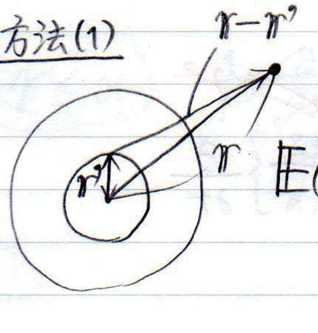
(4) 静電ポテンシャル $\phi(r)$ を求め、これから $\mathbb{E}$ を求める.

$$\begin{cases} \Delta\phi = -\frac{1}{\varepsilon_0}\rho(r) \\ \mathbb{E} = -\mathrm{grad}\,\phi(r) \end{cases}$$

(例) 一様に帯電した球 (半径 a, 電荷 Q)



方法(1)



例えば、$r > a$ のとき

$$\mathbb{E}(r) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{(dr'\cdot r'd\theta\cdot r'\sin\theta d\varphi)}{r^2+r'^2-2rr'\cos\theta}\cdot\cos\theta$$

$\begin{Bmatrix} 0\le r\le a \\ 0\le\theta\le\pi \\ 0\le\varphi\le 2\pi \end{Bmatrix} dxdydz$

方法(2) 既出、grad $\mathbb{E}$

方法(3) (4)と同じなので、飛ばす.

方法(4)

球対称 ⟶ 極座標

(かつ、$\mathbb{E}$ は θ, φ によらない. (r のみの関数))

Δを極座標であらわす.

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \Longleftrightarrow \begin{cases} r = (x^2+y^2+z^2)^{1/2} \\ \tan\theta = (x^2+y^2)^{1/2}/z \\ \tan\varphi = y/x \end{cases}$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\cdot\frac{\partial}{\partial r} + \cancel{\frac{\partial\theta}{\partial x}\cdot\frac{\partial}{\partial\theta}} + \cancel{\frac{\partial\varphi}{\partial x}\cdot\frac{\partial}{\partial\varphi}}$$

cf) 一般に、極座標の場合

$$\boxed{\Delta = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \cancel{\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)} + \cancel{\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2}{\partial\varphi^2}}}$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\cdot\frac{\partial}{\partial r} = \frac{x}{r}\cdot\frac{\partial}{\partial r} \qquad r=\sqrt{x^2+y^2+z^2}$$

$$\therefore \frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{r}\cdot\frac{\partial}{\partial r}\right) = \frac{\partial}{\partial x}\left(x\cdot\frac{1}{r}\cdot\frac{\partial}{\partial r}\right)$$

$$= \frac{1}{r}\frac{\partial}{\partial x} + x\frac{\partial}{\partial x}\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}\right)$$

$$= \frac{1}{r}\cdot\frac{\partial}{\partial x} + \frac{x}{r}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial r}\right) + \frac{x}{r^2}\cdot\frac{\partial r}{\partial x}\cdot\frac{\partial}{\partial r}$$

$$= \frac{x^2}{r^2}\frac{\partial^2}{\partial r^2} + \left\{\frac{1}{r} - \frac{x^2}{r^3}\right\}\frac{\partial}{\partial r}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\underbrace{(x^2+y^2+z^2)}_{r^2}\frac{\partial^2}{\partial r^2} + \left\{\frac{3}{r} - \frac{\overbrace{x^2+y^2+z^2}^{r^2}}{r^3}\right\}\frac{\partial}{\partial r}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\cdot\frac{\partial}{\partial r}$$

$$\begin{cases} r \geq a \text{ で、} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial}{\partial r}\right)\phi = 0 \quad \cdots\cdots ① \quad (\text{Laplaus}) \\ r \leq a \text{ で、} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial}{\partial r}\right)\phi = -\dfrac{1}{\varepsilon_0}\rho \quad \cdots\cdots ② \quad (\text{Poisson}) \end{cases}$$

$$\left(\rho = \frac{Q}{4\pi a^2}\ (一定)\right)$$